$$|\Gamma_g| = \frac{1}{|\Gamma_l|}\frac{|\delta b_1|}{|b_0| + |\delta b_1|},$$

$$|\Gamma_g| = \frac{1}{|\Gamma_l|}\frac{|\delta b_2|}{|b_0| - |\delta b_2|},$$

or if $\Gamma_g$ is small $|\delta b_1| \ll |b_0|$, $|\delta b_1| = |\delta b_2| = |\delta b|$, and

$$|\Gamma_g| \approx \frac{1}{|\Gamma_l|}\frac{|\delta b|}{|b_0|}. \tag{3}$$

An upper limit to $|\Gamma_g|$ due to noninfinite gain of the feedback loop may be obtained by the following experimental technique. First a matched load is connected to the stabilized output and the magnitude of $b_3$ (2) observed. The matched load is then replaced by a sliding short which is adjusted to produce the maximum change in $|b_3|$.

Since $b_2$ is a linear function of $b_3$, the value of $|b_3|$ and $|\delta b_3|$ as thus obtained may be substituted in (3) to obtain an upper limit to the equivalent $|\Gamma_g|$ due to finite gain.

Application of this technique to the stabilizer yielded a corresponding vswr of less than 1.001 which indicates this term is negligible in comparison with

$$\left(S_{22} - \frac{S_{12}S_{23}}{S_{13}}\right).$$

Acknowledgment

The author acknowledges the contributions of R. W. Beatty who proposed the application as a precision attenuator, and thanks him for his valuable criticism in the preparation of the manuscript.

# A Simple Artificial Anisotropic Dielectric Medium*

R. E. COLLIN†

*Summary*—**The anisotropic properties of an infinite stack of thin dielectric sheets separated by another set of thin sheets with a different dielectric constant is investigated. It is shown that the anisotropic properties are brought about because of the two distinct modes of propagation which can exist in such a stacked array of sheets. The limiting forms of the wave solutions and second-order results for the equivalent dielectric constants are given.**

## Introduction

Electromagnetic wave propagation in homogeneous anisotropic dielectric media is well understood and discussed in most text books on optics.[1] At optical frequencies, one has to rely on naturally occurring crystalline media with anisotropic properties. At microwave frequencies where the wavelength is much greater, it is possible to construct artificial dielectric media having either isotropic or anisotropic properties. For example, nonsymmetrical metallic obstacles arranged in a cubical array or symmetrical (also unsymmetrical) obstacles arranged in a noncubical array in a suitable binder will produce an artifical dielectric with anisotropic properties.[2] However, this paper will consider only the anisotropic properties of an infinite stack of dielectric sheets as illustrated in Fig. 1. Each sheet of thickness $t$ and relative dielectric constant $\kappa_a$ is separated by a sheet of thickness $d$ and relative dielectric constant $\kappa_b$. In order to behave essentially as a homogeneous medium, the spacing $S$ must be small in comparison with the wavelength $\lambda_0$ of the radiation. Conditions on $S$ will be given later. Because of the similar disposition of the sheets with respect to the $y$ and $z$ axis, it can be anticipated that this medium will have the same effective dielectric constant along the $y$ and $z$ axis, but a different effective dielectric constant in the $x$ direction and therefore corresponds to a uniaxial crystalline medium.

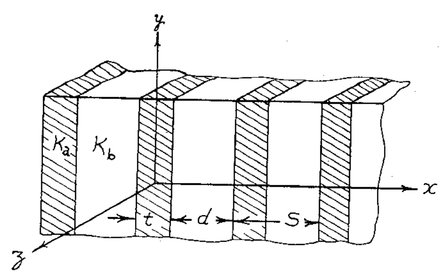


Fig. 1—An artificial anisotropic dielectric medium.

Before considering propagation in this stacked dielectric sheet medium, the theory of wave propagation in a homogeneous uniaxial crystalline medium will be briefly reviewed for later comparison.

* Manuscript received by the PGMTT, August 23, 1957.
† Elec. Eng. Dept., Case Institute of Technology, Cleveland, Ohio; formerly with Canadian Armament Res. and Dev. Estab., Valcartier, P. Q., Can.

[1] G. Joos, "Theoretical Physics," Blackie and Son, Ltd., London, 2nd ed., ch. 19; 1951.

[2] G. Estrin, "The effects of anisotropy in a three-dimensional array of conducting disks," Proc. IRE, vol. 39, pp. 821–826; July, 1951.

## Propagation in an Homogeneous Anisotropic Dielectric

Let the principal dielectric constants be $\kappa_1$, $\kappa_2$, $\kappa_2$ along the principal axes and let these axes coincide with the $x$, $y$, and $z$ axis, respectively. For plane wave propagation, the electric field has the following form

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-j\beta \boldsymbol{n}\cdot\boldsymbol{r}}, \tag{1}$$

where $\boldsymbol{E}_0$ is a constant vector, $\boldsymbol{n}$ is a unit vector along the wave normal, and $\boldsymbol{r}$ is the position vector $\boldsymbol{a}_x x + \boldsymbol{a}_y y + \boldsymbol{a}_z z$. From Maxwell's equations one obtains

$$-j\omega\mu \boldsymbol{H} = \nabla \times \boldsymbol{E} = -j\beta \boldsymbol{n} \times \boldsymbol{E}, \tag{2a}$$

$$j\omega \boldsymbol{D} = \nabla \times \boldsymbol{H} = \frac{j\beta^2}{\omega\mu}[\boldsymbol{E} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{E})]. \tag{2b}$$

Eq. (2b) shows that $\boldsymbol{D}$ is always normal to $\boldsymbol{n}$, but that $\boldsymbol{D}$ and $\boldsymbol{E}$ are not, in general, colinear. If $\boldsymbol{n}\cdot\boldsymbol{E}=0$, then $\boldsymbol{D}$ and $\boldsymbol{E}$ are colinear and must lie in the $yz$ plane. In this case $\beta^2 = \kappa_2 k_0^2$, where $k_0$ is the free-space propagation factor. In general, $\boldsymbol{n}\cdot\boldsymbol{E} \neq 0$ and hence replacing the components of $\boldsymbol{E}$ in the first term on the right-hand side of (2b) by

$$\frac{D_x}{\kappa_1\epsilon_0}, \quad \frac{D_y}{\kappa_2\epsilon_0}, \quad \frac{D_z}{\kappa_2\epsilon_0},$$

and solving for the components of $\boldsymbol{D}$ gives

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{n}\cdot\boldsymbol{E}\frac{\beta^2}{k_0^2}\left[\frac{\boldsymbol{a}_x n_x k_1^2}{\beta^2 - k_1^2} + k_2^2\frac{\boldsymbol{a}_y n_y + \boldsymbol{a}_z n_z}{\beta^2 - k_2^2}\right], \tag{3}$$

where $k_i^2 = \kappa_i k_0^2$. Using the result $\boldsymbol{n}\cdot\boldsymbol{D}=0$ one obtains the following eigenvalue equation from (3).

$$\beta^2 = \frac{k_1^2 k_2^2}{n_x^2 k_1^2 + (1 - n_x^2)k_2^2}. \tag{4}$$

For every direction of the wave normal, except along the optical axis ($x$ axis), two values of the phase velocity are possible. For one mode, $\boldsymbol{E}$ lies in the $yz$ plane and $\beta = k_2$, while for the other mode $\boldsymbol{E}$ has components along all three axis in general and $\beta$ is given by (4). The dielectric displacement $\boldsymbol{D}$ for the two modes are mutually orthogonal and make the same angle with the electric field as the normal $\boldsymbol{n}$ makes with the direction of the Poynting vector.

From the above properties of the two modes of propagation, it is readily seen that they may be derived from a magnetic type and an electric type of Hertzian potential having a single component along the optical axis. The ordinary wave is obtained from the magnetic type Hertzian potential as follows

$$\boldsymbol{E} = -j\omega\mu\nabla \times \boldsymbol{\Pi}_M, \tag{5a}$$

$$\boldsymbol{H} = \nabla \times \nabla \times \boldsymbol{\Pi}_M, \tag{5b}$$

and $\boldsymbol{\Pi}_M$ is a solution of

$$\nabla^2\boldsymbol{\Pi}_M + k_2^2\boldsymbol{\Pi}_M = 0. \tag{5c}$$

The extraordinary wave is obtained from the electric type of Hertzian potential as follows

$$\boldsymbol{E} = k_0^2\boldsymbol{\Pi}_E + \frac{1}{\kappa_2}\nabla\nabla\cdot\boldsymbol{\Pi}_E, \tag{6a}$$

$$\boldsymbol{H} = j\omega\epsilon_0\nabla \times \boldsymbol{\Pi}_E, \tag{6b}$$

and $\boldsymbol{\Pi}_E$ is a solution of

$$\nabla^2\boldsymbol{\Pi}_E + k_1^2\boldsymbol{\Pi}_E + \frac{\kappa_1 - \kappa_2}{\kappa_2}\frac{\partial^2\boldsymbol{\Pi}_E}{\partial x^2} = 0. \tag{6c}$$

When $\boldsymbol{n} = \boldsymbol{a}_x n_x$; *i.e.*, coincides with the optical axis, the solutions generated by $\boldsymbol{\Pi}_M$ and $\boldsymbol{\Pi}_E$ vanish. However, the solution in this case is a simple plane wave transverse to the $x$ axis and with a propagation constant $k_2$.

## Propagation in a Stacked Dielectric Sheet Medium

If the $x$ axis is considered as the axis of propagation in the medium illustrated in Fig. 1, then the modes of propagation are the $E$ and $H$ modes. These modes are derivable from the electric and magnetic Hertzian potentials, respectively, as follows[3]

for $E$ modes,

$$\boldsymbol{E} = \frac{1}{\kappa(x)}\nabla \times \nabla \times \boldsymbol{\Pi}_E, \tag{7a}$$

$$\boldsymbol{H} = j\omega\epsilon_0\nabla \times \boldsymbol{\Pi}_E, \tag{7b}$$

$$\nabla^2\boldsymbol{\Pi}_E - \frac{1}{\kappa(x)}\nabla\kappa(x)\nabla\cdot\boldsymbol{\Pi}_E + \kappa(x)k_0^2\boldsymbol{\Pi}_E = 0, \tag{7c}$$

while for $H$ modes

$$\boldsymbol{E} = -j\omega\mu\nabla \times \boldsymbol{\Pi}_M, \tag{8a}$$

$$\boldsymbol{H} = \nabla \times \nabla \times \boldsymbol{\Pi}_M, \tag{8b}$$

$$\nabla^2\boldsymbol{\Pi}_M + \kappa(x)k_0^2\boldsymbol{\Pi}_M = 0, \tag{8c}$$

where

$$\kappa(x) = \begin{cases} \kappa_a, & mS - \dfrac{t}{2} \le x \le mS + \dfrac{t}{2}, \\ \kappa_b, & mS + \dfrac{t}{2} \le x \le (m+1)S - \dfrac{t}{2}, \end{cases}$$

and $m$ is any integer. $\boldsymbol{\Pi}_E$ and $\boldsymbol{\Pi}_M$ have components along the $x$ axis only. In view of the periodic nature of $\kappa(x)$, the solutions for $\boldsymbol{\Pi}_E$ and $\boldsymbol{\Pi}_M$ are of the form[4]

$$\boldsymbol{\Pi}_E = \boldsymbol{a}_x e^{-j\beta\boldsymbol{n}\cdot\boldsymbol{r}}F_E(x), \tag{9a}$$

$$\boldsymbol{\Pi}_M = \boldsymbol{a}_x e^{-j\beta\boldsymbol{n}\cdot\boldsymbol{r}}F_M(x), \tag{9b}$$

[3] R. E. Collin and R. M. Vaillancourt, "Application of Rayleigh-Ritz method to dielectric steps in waveguides," IRE Trans., vol. MTT-5, pp. 177–184; July, 1957.

[4] R. E. Collin, "Reflection and transmission at a slotted dielectric interface," *Can. J. Phys.*, vol. 34, pp. 398–411; April, 1956.

where $F_E$ and $F_M$ are periodic functions of $x$ with a period $S$. The boundary conditions are continuity of

$$\Pi_M, \frac{\partial \Pi_M}{\partial x}, \frac{1}{\kappa(x)} \frac{\partial \Pi_E}{\partial x}, \text{ and } \Pi_E.$$

The fundamental solutions for $\Pi_E$ and $\Pi_M$ are readily constructed from the independent even and odd solutions of the wave equation. The details are given by Collin[4] for a case corresponding to a solution for $\Pi_M$. A similar procedure may be used for $\Pi_E$ but account must be taken of the modified boundary condition on $\partial\Pi_E/\partial x$. The solutions for the general case will not be given here, but rather the limiting forms of these solutions as $k_0S$ approaches zero. It is found that

$$\mathbf{\Pi}_E \to \mathbf{a}_x e^{-j\beta \mathbf{n}\cdot\mathbf{r}} F_E(x), \tag{10a}$$

where

$$F_E(x) = F_E(x + mS)$$
$$= \begin{cases} 1 - j\beta n_x \dfrac{\kappa_a - \kappa_b}{\kappa_2}\left(1 - \dfrac{t}{S}\right)x, & -\dfrac{t}{2} \le x \le \dfrac{t}{2}, \\ 1 - j\beta n_x \dfrac{\kappa_a - \kappa_b}{\kappa_2}\dfrac{t}{S}\left(\dfrac{S}{2} - x\right), & \dfrac{t}{2} \le x \le S - \dfrac{t}{2}, \end{cases} \tag{10b}$$

$$\beta^2 = \frac{k_1^2 k_2^2}{n_x^2 k_1^2 + (1 - n_x^2) k_2^2}, \tag{10c}$$

$$\kappa_1 = \left[\frac{1}{\kappa_b} - \frac{\kappa_a - \kappa_b}{\kappa_a \kappa_b}\frac{t}{S}\right]^{-1}, \tag{10d}$$

$$\kappa_2 = \kappa_b + (\kappa_a - \kappa_b)\frac{t}{S}, \tag{10e}$$

and that

$$\mathbf{\Pi}_M \to \mathbf{a}_x e^{-jk_2 \mathbf{n}\cdot\mathbf{r}}, \tag{11}$$

where $k_2^2 = \kappa_2 k_0^2$ and $\kappa_2$ is given by (10e). The limiting solution for $\Pi_M$ is thus a plane wave corresponding to the ordinary wave in the equivalent homogeneous anisotropic dielectric. The limiting solution for $\Pi_E$ is essentially also a plane wave. It is modified by a small oscillating phase function. The field components are derivable from $\Pi_E$ by means of (7a)–(7c). If these field components, apart from the propagation factor $e^{-j\beta n\cdot r}$, are averaged over one period along the $x$ axis, one finds that the average values are the same as would be obtained for a homogeneous anistotropic dielectric with relative dielectric constants given by (10d) and (10e). Thus, in the limit as $k_0S$ approaches zero the stacked dielectric sheet medium becomes identical with a homogeneous anisotropic dielectric. The maximum anisotropic effect is obtained when

$$\frac{t}{S} = 0.50, \tag{12a}$$

and for this value of $t/S$ the ratio of the two dielectric constants is

$$\frac{\kappa_2}{\kappa_1} = \frac{(\kappa_a + \kappa_b)^2}{4\kappa_a\kappa_b}. \tag{12b}$$

This ratio is not critically dependent on $t/S$. As $\kappa_a$ becomes much larger than $\kappa_b$, this ratio approaches $\kappa_a/4\kappa_b$. For polystyrene sheets separated by polyfoam sheets and the optimum value of $t/S$, $\kappa_2/\kappa_1 = 1.22$.

Even for large values of $k_0S$ this medium has anisotropic properties, because of the existence of two distinct modes of propagation with different phase velocities. In fact, for arbitrary large values of $k_0S$, there are many modes of propagation. However, provided that $k_0S$ satisfies the following inequality, only the two fundamental modes will propagate with a wave normal having real direction cosines,

$$k_0S < \frac{2\pi}{\sqrt{\kappa}}, \tag{13}$$

where $\kappa$ is the largest value of $\kappa_a$ or $\kappa_b$. This relation is obtained by imposing the condition that no higher order mode having the required periodic behavior should propagate in a homogeneous material with a dielectric constant $\kappa$.

The plane wave approximations for $\Pi_E$ and $\Pi_M$ are accurate for that range of $k_0S$ for which the equivalent dielectric constants are well approximated by (10d) and (10e). For general values of $k_0S$, second-order approximations for the equivalent dielectric constants are,

$$\kappa_1' = \kappa_1 + \frac{\kappa_b^2[(P_{11} + P_{00})^2 - 4\Delta]}{4\Delta P_{00}(2P_{00} - P_{11})}\frac{S^2}{\lambda_0^2}, \tag{14a}$$

where

$$\Delta = P_{00}P_{11} - P_{01}^2,$$

$$P_{00} = 1 - \frac{\kappa_a - \kappa_b}{\kappa_a}\frac{t}{S},$$

$$P_{01} = -\frac{\kappa_a - \kappa_b}{\kappa_a}\frac{\sqrt{2}}{\pi}\sin\pi\frac{t}{S},$$

$$P_{11} = P_{00} - \frac{\kappa_a - \kappa_b}{\kappa_a}\frac{1}{2\pi}\sin 2\pi\frac{t}{S},$$

$$\kappa_2' = \kappa_2 + \frac{(\kappa_a - \kappa_b)^2}{16\pi^2} \cdot \left[32\sin^2\pi\frac{t}{S} + \sin^2 2\pi\frac{t}{S}\right]\frac{S^2}{\lambda_0^2}. \tag{14b}$$

These two expressions were obtained by an application of the Rayleigh-Ritz method.[3] Propagation in the $yz$ plane was considered and a two term approximation used for $\Pi_E$ and $\Pi_M$. The equivalent dielectric constants were defined so as to give the same phase velocity in an equivalent homogeneous medium as was actually found

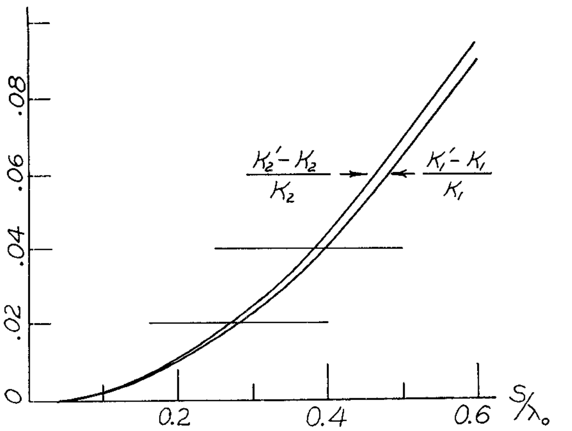


Fig. 2—Deviation of equivalent dielectric constants from static values. $\kappa_a=2.56$, $\kappa_b=1.03$.

for the stacked dielectric sheet medium. Quadratic expressions for $\kappa_1'$ and $\kappa_2'$ were obtained and (14a) and (14b) were derived by expanding the square roots and retaining only the lowest power term in $S^2/\lambda_0^2$. It is seen that $\kappa_1'$ and $\kappa_2'$ reduce to $\kappa_1$ and $\kappa_2$, respectively, whenever $k_0S$ becomes negligible. Fig. 2 is a plot of

$$\frac{\kappa_1' - \kappa_1}{\kappa_1} \quad \text{and} \quad \frac{\kappa_2' - \kappa_2}{\kappa_2}$$

as a function of $S/\lambda_0$ for $\kappa_a=2.56$, $\kappa_b=1.03$, and the optimum value of $t/S$. For this case, the plane wave approximation is satisfactory for $S$ less than about $0.3\lambda_0$.

This artificial anisotropic dielectric has been used as the basis of a broad-band quarter-wave plate and circular polarizer,[5] as well as to improve the impedance characteristics of a periodically loaded waveguide for linear accelerator use.[6] Another use for this medium is for a microwave version of a Nicol prism for analyzing an arbitrary polarized wave into its basic components.

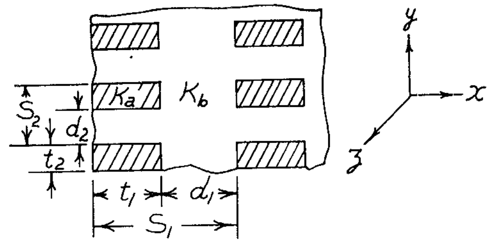


Fig. 3—A three-dimensional artificial anisotropic dielectric medium.

An artificial dielectric medium having three dimensional anisotropic properties can be obtained by making the structure periodic along two directions as in Fig. 3. The principal values of the dielectric constants for this medium can readily be evaluated when the periods $S_1$ and $S_2$ are small compared with the wavelength, so that the static values may be used. When the fringing fields around the dielectric prisms are neglected, it is found that

$$\kappa_x = \frac{S_1\kappa_b(t_2\kappa_b + d_2\kappa_a)}{\kappa_b(S_2d_1 + t_1t_2) + t_1d_2\kappa_a}, \tag{15a}$$

$$\kappa_y = \frac{S_2\kappa_b(t_1\kappa_b + d_1\kappa_a)}{\kappa_b(S_1d_2 + t_1t_2) + t_2d_1\kappa_a}, \tag{15b}$$

$$\kappa_z = \kappa_b + (\kappa_a - \kappa_b)\frac{d_1d_2}{S_1S_2}. \tag{15c}$$

## Conclusion

The stacked dielectric sheet medium has been shown to behave like a homogeneous anisotropic dielectric when the spacing between sheets is small compared with the wavelength. Furthermore, the principal values of the dielectric constants are given to sufficient accuracy by their static values. For larger spacings, the equivalent dielectric constants are a few per cent greater than the static values, but the anisotropic properties are retained because of the existence of two distinct modes of propagation in the medium.

[5] H. S. Kirschbaum and S. Chen, "A method of producing broad-band circular polarization employing an anisotropic dielectric," IRE Trans., vol. MTT-5, pp. 199–203; July, 1957.

[6] R. B. R. Harvie, *et al.*, "A theoretical and experimental investigation of anisotropic dielectric loaded linear electron accelerators," *Proc. IEE (London)*, Pt. B, vol. 104, pp. 273–292; May, 1957.